MATH 300: HWK #2 Solutions

Section 1.4: #5c

Claim: Let $x, y \in \mathbb{Z}$. If x and y are even, then xy is divisible by 4.

Proof: Let $x, y \in \mathbb{Z}$ be even. Then $\exists\ k, \ell \in \mathbb{Z}$ such that

$$x = 2k, \quad y = 2\ell.$$

Thus, $xy = (2k)(2\ell) = 4(k\ell)$.

Since $k\ell$ is an integer, this means $4 \mid xy$. ∎

Section 1.4: #7d

Claim: Let $a \in \mathbb{Z}$. Then $a(a+1)$ is even.

Proof: Let $a \in \mathbb{Z}$. Then a is either even or odd.

If a is even then $\exists\ k \in \mathbb{Z}$ such that $a = 2k$, so $a(a+1) = 2k(2k+1) = 2[2k^2 + k]$, which means $a(a+1)$ is even.

If a is odd then $\exists\ \ell \in \mathbb{Z}$ such that $a = 2k+1$, so $a(a+1) = (2k+1)(2k+2) = 2[(2k+1)(k+1)]$, which means $a(a+1)$ is even.

Thus, in all cases $a(a+1)$ is even. ∎

Section 1.5: #3c

Claim: Let $x \in \mathbb{Z}$. If x^2 is not divisible by 4, then x is odd.

Proof: We proceed by contrapositive. Suppose that $x \in \mathbb{Z}$ and x is not odd, so it is even. Thus, $\exists\, k \in \mathbb{Z}$ such that $x = 2k$, so

$$x^2 = (2k)^2 = 4k^2$$

which is divisible by 4. ∎

Section 1.5: #7b

Claim: Suppose $a, b \in \mathbb{Z}$ with $a, b > 0$. Then $(a+1) \mid b$ and $b \mid (b+3)$ if and only if $a = 2, b = 3$.

Proof: First assume $a = 2$ and $b = 3$. Then $a + 1 = 3$ so $(a+1) \mid b$ and $b + 3 = 6$ so $b \mid b+3$.

To prove the converse, assume $a, b \in \mathbb{Z}$, $a, b > 0$, are such that ① $a+1 \mid b$ and ② $b \mid b+3$

By ② $\exists\, k \in \mathbb{Z}$ such that $bk = b + 3$ so $b(k-1) = 3$, which means $b \mid 3$ and $b > 0$, so $b = 1$ or $b = 3$. But if $b = 1$ then $a+1 \mid 1 \Rightarrow a+1 = 1$ or $a+1 = -1$, both of which are impossible for $a > 0$. Thus, $b = 3$. Now $a+1 \mid b$ means a is one of $-4, -2, 0,$ or 2. The only possibility is $a = 2$. Thus $a = 2, b = 3$. ∎